Relation and Function

1. Prove that the binary operation defined on set N, given by $a * b = \frac{a+b}{2}$ for all $a, b \varepsilon N$, is commutative? Is the above binary operation associative?
2. If $f(x)$ is an invertible function, find the inverse of $f(x) = \frac{3x-2}{5}$
3. Let T be the set of all triangles in a plane with R as relation in T given by $R = \{(T1, T2) : T1 \cong T2\}$. Show that R is an equivalence relation.
4. Prove that the relation R on the set A = {1, 2, 3, 4, 5} given by $R = \{(a, b) : |a - b| \text{ is even }\}$, is an equivalence relation
5. Let $f : N \to N$ be defined by

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if is odd,} \\ \frac{n}{2} & \text{if is even for all } n \varepsilon N \end{cases}$$

   Find whether the function f is bijective
6. Show that the relation R in the set of real numbers, defined as $R = \{(a, b) : a \leq b^2\}$ is neither reflexive, nor symmetric, nor transitive.
7. Let Z be the set of all integers and R be the relation on Z defined as $R = \{(a, b)$; $a, b \varepsilon Z$, and $(a - b)$ is divisible by 5.} Prove that R is an equivalence relation.
8. If $f : R \to R$ be defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$.
9. Show that the relation R in the set $A = \{x \varepsilon Z : 0 \leq x \leq 12\}$ given by $R = \{(a, b) : a, b \varepsilon Z, |a - b|$ is divisible by 4} is an equivalence relation. Find the set of all elements related to 1.
10. Show that the relation S defined on the set N X N by(a, b) $S(c, d) \to a + d = b + c$ *is equivalence*.
11. Consider $f : R \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$
12. Let $A = N$ X N and * be a binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that * is commutative and associative. Also, find the identity element for * on A, if any.
13. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2,5), (3, 6)} be a function from A to B. State whether f is one-one or not.
14. Let $f : R \to R$ be defined as $f(x) = 10x + 7$. Find the function $g : R \to R$ such that gof = fog = IR .
15. A binary operation * on the set {0, 1, 2, 3, 4, 5} is defined as: $a*b = a+b$ if $a+b<6$,
    $a*b = a+b-6$ if $a+b \geq 6$
    Show that zero is the identity for this operation and each element 'a' of the set is invertible with $6-a$, being the inverse of 'a'.
16. Consider the binary operation* on the set {1, 2, 3, 4, 5} defined by a * b = min. {a, b}. Write the operation table of the operation *.
17. If $f : R \to R$ and $g : R \Rightarrow R$ are given by $f(x) = \sin x$ and $g(x) = 5x^2$, find $gof(x)$.
18. If $f : R \to R$ is defined by $f(x) = 3x + 2$, define $f[f(x)]$.
19. Consider $f : R_+ \to [4, \infty]$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse (f^{-1}) of f given by $f^{-1}(y) = (\sqrt{y-4})$, where R_+ is the set of all non-negative real numbers.
20. Write fog, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = |x|$ and $g(x) = |5x - 2|$
21. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x-3}$ Show that f is one-one and onto and hence find f^{-1}
22. Show that f : N ⇒ N, given by

$$f(x) = \begin{cases} x+1 & \text{if x is odd ,} \\ x-1 & \text{if is even} \end{cases}$$

   is both one-one and onto.
23. Consider the binary operations $* : R \times R \to$ R and o : $R \times R \to R$ defined as $a * b = |a - b|$ and aob = a for all $a, b \varepsilon R$. Show that '*' is commutative but not associative, 'o' is associative but not commutative.
24. If the binary operation * on the set Z of integers is defined by $a * b = a + b - 5$, then write the identity element for the operation * in Z.

25. If $f(x) = \{ \frac{4x+3}{6x-4}$, $x \neq 2/3\}$. Show that $fof(x) = x$ for all $x \neq 2/3$. What is the inverse of f?

INVERSE TRIGONOMETRIC FUNCTION

1. Prove that: $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

2. Prove that: $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left[\frac{1}{3}\right] + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

3.Prove that: $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right)$

4.Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

5. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x, \frac{-1}{\sqrt{2}} \leq x \leq 1$

6.Prove the following: $\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

7. Prove that: $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

8.Prove that: $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$.

9.Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

10. Find the value of x if $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

11. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in (0, \frac{\pi}{4})$

12. Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, (x > 0)$

13. Express in the simplest form $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x \neq 0$

14. Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$.

15. Prove $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0,1)$

16. Prove that $\text{Cos}(\tan^{-1}(\sin(\cot^{-1}x))) = \left(\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right)$

17. Prove that $\sin(2\tan^{-1}\left(\frac{1}{3}\right) + \cos(\tan^{-1}2\sqrt{2}) = \frac{14}{15}$

18. $\cot\ (\frac{\pi}{4} - 2\cot^{-1}3) = 7$

19. Find the value of x, If $\sin(\cot^{-1}(x+1)) = \cos(\tan^{-1}x)$

20. Find the value of x If $\tan^{-1}\left(\frac{1}{1+1.2}\right) + \tan^{-1}\left(\frac{1}{1+2.3}\right) + \ldots.\tan^{-1}\left(\frac{1}{1+n.(n+1)}\right) = \tan^{-1}x$

DETERMINANT

Type 1 problems:- Problems in which we have any row or column has entry 1 only or by taking common easily we can get 1,1,1 in a row or column

1. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ = (x – y)(y –z)(z – x)

2. $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix}$ $=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

3. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$ = (a - b)(b - c)(c - a)(a + b + c)

4. $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ = xyz(x – y)(y – z)(z – x)

Type 2 problems:- Problems in which we can apply $R_1 \rightarrow R_1 + R_2 + R_3$ or $C_1 \rightarrow C_1 + C_2 + C_3$ get a common term in a row or column .

5. $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$ $= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

6. $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ = $(5x + 4)(x – 4)^2$

7. $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a. \end{vmatrix}$ = $9(a + b) b^2$

8. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ = $(a + b + c)^3$

9. $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

10. $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z)$

Type 3 problems:- Problems in which we cannot apply $R_1 \rightarrow R_1 + R_2 + R_3$ or $C_1 \rightarrow C_1 + C_2 + C_3$ directly but after applying some operations (multiplying and dividing by a,b,c and taking common a,b,c we can reduce the problem into type 2 problems

11. $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

12. Using the Properties of Determinants Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

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13. Using the Properties of Determinants Prove that $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$

Type 4 problems:- Problems in which we have to apply **specific operations** only in order solve the problem easily otherwise it may become lengthy and difficult to solve.

14. Using the properties of Determinants, show that $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$

15. Using the properties of Determinants, show that $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2)$

16. Using the properties of Determinants, show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

17. Using the Properties of Determinants Prove that $\begin{vmatrix} (b+c)^2 & ab & ac \\ ab & (a+c)^2 & bc \\ ca & cb & (a+b)^2 \end{vmatrix}$ = 1 +a^2+b^2+c^2

Type 5 problems:- Solve for x problems.

18. Show that x = 2 is a root of equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$ and solve it completely.

19. Using properties of determinants, solve the following for x: $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

20. Using properties of determinants, solve the following for x: $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$

Type 6 problems:- Problems based on property 7

21. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$

22. If $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ and x,y,z are all different, Prove that xyz = -1

Type 7 problems:- To find the value of determinant or prove without expansion.

23. Without expansion prove that $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

24. Without expansion evaluate $\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$

25. Without expansion evaluate $\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ \sqrt{115}+3 & \sqrt{5} & 5 \end{vmatrix}$

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MATRIX

1. If $A=\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}$; Then show that A^3-23A-40$\boldsymbol{I}$=0
2. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$; verify that $A^2 - 4A - 5I = 0$. Hence find A^{-1}
3. $If, A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix} and, I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ Find 'k' so that A^2=kA-2I.
4. By using elementary operations find the inverse of the matrix i) $\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}$ ii) $\begin{bmatrix}1 & 3 & -2\\ -3 & 0 & -1\\ 2 & 1 & 0\end{bmatrix}$
5. There are three families. First family consists of 2 male members, 4 female members and 3 children. Second family consists of 3 male members, 3 female members and 2 children. Third family consists of 2 male members, 2 female members and 5 children. Male member earns Rs 500 per day and spends Rs 300 per day. Female member earns Rs 400 per day and spends Rs 250 per day child member spends Rs 40 per day. Find the money each family saves per day using matrices? What is the necessity of saving in the family?
6. If $f(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$ prove that f(x).f(y) =f(x +y)
7. If A = $\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$ Prove that $A^n = \begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$ for all natural number
8. If A = $\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}$, B = $\begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix}$ and C = $\begin{bmatrix}2 & 5\\ 3 & 8\end{bmatrix}$ find D such that CD – AB = O
9. Find a,b such that $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}$ is a matrix which satisfies $A'A = 9I_3$.
10. Express $A=\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}$ as a sum of a symmetric and a skew symmetric matrix.
11. If A = $\begin{bmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{bmatrix} A=\begin{bmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{bmatrix}$
12. If $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 4\\ 0 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & -4\\ 2 & -1 & 5\end{bmatrix}$ verify that AB = BA 6I hence solve the linear equations

    x – y = 3,2x + 3y + 4z = 17 and y + 2z = 7

13. Using matrices, solve the following system of equation:

    $2x-3y+5z=11$, $3x+2y-4z=-5$, $x+y-2z=-3$
14. Using matrix method, solve the following system of equations :

    3x-2y+3z=8. , 2x+y- z=1, 4x-3y+2z=4.
15. An amount of Rs 600 crores is spent by the government in three schemes. Scheme A is for saving girl child from the cruel parents who don't want girl child and get the abortion before her birth. Scheme B is

for saving of newlywed girls from death due to dowry. Scheme C is planning for good health for senior citizen. Now twice the amount spent on Scheme C together with amount spent on Scheme A is Rs 700 crores. And three times the amount spent on Scheme A together with amount spent on Scheme B and Scheme C is Rs 1200 crores. Find the amount spent on each Scheme using matrices? **What is the importance of saving girl child from the cruel parents who don't want girl child and get the abortion before her birth?**

16. A school has to reward the students participating in co-curricular activities (Category I) ,with 100% attendance (Category II) and class toppers (Category III) in a function. The sum of the numbers of all the three category students is 6. If we multiply the number of students of category III by 2 and added to the number of students of category I , we get 7. By adding the number of students of category II and III with three times the number of students of category I we get 12.Form the matrix equation and solve it. ***Do you think the school should add one more category to motivate the students for cleanliness? Give your idea in brief.***